PG – 222

IV Semester M.Sc. Degree Examination, June 2016
(CBCS)
MATHEMATICS
M401 T : Measure and Integration

Time : 3 Hours Max. Marks : 70

Instructions : *i) Answer **any five full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) Show that the Lebesgue outer measure of an interval is its length. 7

b) Let $\{E_n\}$ be a countable collection of sets. Then prove that
$m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n)$. 7

2. a) Define G_δ-sets. Let E be any set. Then prove that

i) Given $\varepsilon > 0$, there exists an open set $O \supset E$ such that $m^*(O) < m^*(E) + \varepsilon$.

ii) There exists a G_δ-set $G \supset E$ such that $m^*(E) = m^*(G)$. 7

b) Define Lebesgue measurable set. If E has the outer measure zero, then prove that E is measurable, hence show that every subset of E is measurable. 7

3. a) If f and g are two measurable functions defined on the same domain then prove that f + g, fg, f^2 and |f| are measurable. 8

b) Let f and g be two functions defined on the same domain E such that f = g a.e. and g is measurable, then prove that f is measurable. 6

4. a) State and prove Egoroff's theorem. 8

b) Let $\{f_n\}$ be a sequence of measurable functions which converges to f a.e. on E, then prove that $\{f_n\} \to$ f on E. 6

P.T.O.

5. a) Define a simple function. Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then prove the following :

   i) $\int a\phi + b\psi = a\int\phi + b\int\psi$, a, b $\in \mathbb{R}$.

   ii) If $\phi \geq \psi$ a.e. then $\int\phi \geq \int\psi$. 4

   b) Prove that a bounded function defined on a measurable set E of finite measure is Lebesgue integrable if and only if it is measurable. 6

   c) If f and g are non-negative measurable functions then prove :

   i) $\int_E cf = c\int_E f$, $c \in \mathbb{R}$. ii) $\int_E f + g = \int_E f + \int_E g$ 4

6. a) If f is an integrable function on a measurable set E, then prove that |f| is also integrable. 4

   b) State and prove Lebesgue's dominated convergence theorem. 6

   c) Define a function of bounded variation. Prove that a monotonic function on [a, b] is of bounded variation. 4

7. a) Define a Vitali cover. Establish the Vitali covering lemma. 8

   b) Let f be an increasing real-valued function defined on [a, b]. Then prove that f is differentiable a.e. Also prove that f' is measurable and $\int_a^b f'(x)dx \leq f(b) - f(a)$. 6

8. a) Prove that L^p spaces are normed linear spaces for $1 \leq p < \infty$. 7

   b) State and prove Riesz-Fischer theorem. 7

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